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Invariants of t-structures and classification of nullity classes

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Abstract

We construct an invariant of t-structures on the derived category of a commutative noetherian ring. This invariant is complete when restricting to the category of complexes with finitely generated bounded homology, and also gives a classification of nullity classes with the same restriction. On the full derived category of $\mathbb{Z}$ we show that the class of distinct t-structures do not form a set.

MSC: 18E30; 18E35

Keywords: Nullity class; t-structures; Derived categories

1. Introduction

A t-structure on a triangulated category generalizes the idea of truncating the homology of a chain complex above a specified degree, and was introduced by Beilinson, Bernstein and Deligne in [3]. This paper constructs an invariant of t-structures on $D(R)$, the derived category of a commutative noetherian ring R. When restricting attention to the complexes with finitely generated bounded homology $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, we show that this invariant is complete (Theorem 5.6), in other words two different t-structures have different values of the invariant. We also show this invariant classifies slightly weaker structures called nullity classes.

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The Postnikov section functor P_n, which kills all homotopy groups $\pi_i(X) = [S^i, X]$ above dimension n, provides a notion of truncation in topological spaces. Originating in the work of Bousfield [7] and Dror Farjoun [9], for any space E there is a more general truncation functor P_E which kills all maps from E and its suspensions so that $[\Sigma^i E, P_E(X)] = 0$. It can be defined to be the universal functor with this property. As a special case we get the Postnikov section, $P_n = P_{S^{n+1}}$.

An example of a nullity class is $\overline{C(E)}$ which is the class of objects such that $P_E(X) = 0$. More generally in a triangulated category a nullity class is a full subcategory closed under arbitrary coproducts, (positive) suspensions and extensions. Keller and Vossieck [15] showed in any triangulated category $\mathbf{T}$ that t-structures are in bijection with nullity classes $\mathcal{A} \subset \mathbf{T}$ admitting a right adjoint to the inclusion functor (Theorem 2.11). They called such classes aisles. Hirschhorn [11] has constructed the Bousfield localization with respect to a map in a model category framework, and for any object E, localizing with respect to $E \to *$ gives the functor P_E. Alonso Tarrio, Jeremias Lopez and Souto Salorio [1] constructed the functor P_E in the category $D(R)$, and also showed that $\overline{C(E)}$ is an aisle. This gives a general way to construct t-structures. Since the isomorphism classes of objects in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ form a set, it is not hard see that when restricted to $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ this gives all the nullity classes and all the t-structures; Corollaries 6.13 and 5.7 give one way to prove this.

Our approach has its roots in topology and thick subcategories. If a nullity class in $D(R)$ is also closed under desuspension then it is a localizing subcategory, an infinite version of a thick subcategory. The thick subcategories of p-local finite spectra were classified by Hopkins and Smith [13] in terms of an invariant called type. Bousfield [8] used their classification to classify nullity classes of p-torsion finite suspension spaces. Bousfield's classification is in terms of two things: type, which tells us which thick subcategory the class generates stably and connectivity, which tells us where the class starts. Using ideas from the classification for spectra, Hopkins [12] and Neeman [17] classified thick subcategories of the perfect complexes $D_{\mathrm{perf}}(R)$ for noetherian rings R by subsets of Spec R closed under specialization (see 2.1). The invariant is given by taking the support of the object. Neeman [17] proved the analogous result in $D(R)$ where localizing subcategories are classified by all subsets of Spec R, and Thomason [21] made an extension of these results to schemes.

1.1. An invariant of nullity classes and t-structures and the main results

Starting with a nullity class $\mathcal{A}$ in $D(R)$ or $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, for example the aisle of a t-structure, we associate a function (see 4.4):

$$\phi(\mathcal{A})\colon \mathbb{Z} \to \bigl\{\text{Subsets of } \operatorname{Spec}(R) \text{ closed under specialization}\bigr\},$$

whose value at n, $\phi(\mathcal{A})(n)$ can be thought of in the following way. Truncate $\mathcal{A}$ above n with the standard truncation to get $\tau^{\geqslant -n}\mathcal{A}$. Next take the thick subcategory generated by $\tau^{\geqslant -n}\mathcal{A}$ and apply the correspondence of Hopkins–Neeman, in other words take supports. This subset of $\operatorname{Spec}(R)$ is the value $\phi(\mathcal{A})(n)$. Since, as in Bousfield's result, we cannot desuspend, we have to prescribe at what level the aisle of the t-structure starts and there is some choice of when different primes can start. If $p \in \phi(\mathcal{A})(n)$ it means that that prime has already been included at level n. Since $\mathcal{A}$ is closed under suspensions, we see that $\phi(\mathcal{A})$ must be increasing. In this way we have been motivated by applying Bousfield's philosophy to the Hopkins–Neeman result, and from Theorem 6.12 we get:

Theorem A. *If* $\dim(R)$ *is finite, then* ϕ *is an order preserving bijection between nullity classes in* $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ *and increasing functions from* $\mathbb{Z}$ *to subsets of* $\operatorname{Spec} R$ *closed under specialization.*

As pointed out to me by Halvard Faulk, such an increasing function contains exactly the same information as a traditional monotone perversity function (see 4.2). Since all aisles are nullity classes (Theorem 2.14) when $\dim R$ is finite, Theorem A also implies ϕ is a complete invariant of t-structures in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$. More generally, in the case $\dim R$ is not necessarily finite we also give a direct, and considerably shorter, proof that ϕ is a complete invariant (Theorem 5.6).

Theorem B. ϕ *induces an order preserving injection from aisles in* $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ *to increasing functions from* $\mathbb{Z}$ *to subsets of* $\operatorname{Spec} R$ *closed under specialization.*

Although in $D(R)$ all the $\overline{C(E)}$ are aisles, their restriction to $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, $\overline{C(E)} \cap D^{\mathrm{b}}_{\mathrm{fg}}(R)$ may not be an aisle. The problem is that for $M \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$, $P_E(M)$ which is one of the truncations of the t-structure, may no longer be in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$. However we have some understanding of the image of ϕ, and in fact the primes must be added in a very controlled way for the nullity class to have a chance of being an aisle. We get Theorem 7.9:

Theorem C. *Suppose* $\mathcal{A} \subset D^{\mathrm{b}}_{\mathrm{fg}}(R)$ *is an aisle. If* $p' \in \phi(\mathcal{A})(n)$ *and* p *is maximal under* p'*, then* $p \in \phi(\mathcal{A})(n+1)$.

We conjecture (Conjecture 7.10) the converse of the theorem: all nullity classes satisfying the condition are aisles. Proving the conjecture would complete the classification of t-structures in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$. We call a function satisfying the growth condition in the theorem *comonotone* (see 4.2). This agrees with the definition of comonotone perversity given by Bezrukavnikov [4, Definition 3]. In case R has a dualizing complex then the conjecture is the special case of the result of Deligne from the same paper [4, Theorem 1] when G is the trivial group and the scheme is $\operatorname{Spec}(R)$. Alonso, Jeremias and Saorin [2] have proved the conjecture when R has a pointwise dualizing complex. The conjecture is still open when R does not have a dualizing complex. If $\dim(R)$ is finite the conjecture could be rephrased as saying that ϕ should induce a bijection between aisles and monotone, comonotone perversities. It is worth remarking at this point that the conjecture would be false if we were to work with perfect complexes (see Example 7.11), so $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ seems to be the right place to work when taking this point of view.

In the last section of the paper we use examples of Shelah [19] to show that there is not a set, but rather a proper class of t-structures in $D(\mathbb{Z})$ (Corollary 8.4):

Theorem D. *The class of t-structures, and hence also the class of nullity classes, in* $D(\mathbb{Z})$ *do not form a set.*

This not only shows a strong contrast to what happens with localizing subcategories in $D(R)$, but also shows that classifying the t-structures in $D(R)$ is probably not feasible. These examples can be transported to the topological setting to show there exists no set of nullity classes in spectra or topological spaces and no set of t-structures in the triangulated category of spectra.

Next we give a short description of the contents of each section, more details and comments can be found at the start of some sections. Section 2 gives some background from ring theory and derived categories and also introduces aisles, nullity classes and the nullification functor P_E. It sets some of the notation for the paper so should probably be at least skimmed before reading the rest of the paper. Section 3 proves some properties of nullity classes and P_E that are well known in the topological setting. It can easily be skipped and referred to when needed. Section 4 defines the invariant ϕ and another function N which is the inverse of ϕ. It starts with again giving a concise summary of the results of the paper that is worth looking at. In Section 5 we give a short proof that ϕ is a complete invariant when restricted to aisles in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$. The most technical part of the paper is Section 6 where the classification of the nullity classes in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ is completed. In Section 7 we get restrictions on the image of ϕ when restricted to aisles. Section 8 constructs the examples in $D(\mathbb{Z})$ that show the aisles, and hence the nullity classes do not form a set.

2. Background and notation

Throughout this paper we let R be a commutative noetherian ring.

2.1. Ring theory and associated primes

Recall that $\operatorname{Spec} R$ is the set of primes of R. If $U \subset \operatorname{Spec} R$, then $\overline{U}$ denotes the closure of U under specialization. That is,

$$\overline{U} = \{p \in \operatorname{Spec} R \mid \exists q \in U,\ q \subset p\}.$$

We will repeatedly use the concepts of associated prime and support.

Definition 2.1. For an R-module M, $\operatorname{Ass} M$ denotes the set of associated primes of M. So $p \in \operatorname{Ass} M$, if p is the annihilator of some element of M.

$$\operatorname{Ass} M = \{p \in \operatorname{Spec} R \mid \exists x \in M,\ \operatorname{ann} x = p\},$$

where for $x \in M$, $\operatorname{ann} x$ denotes the annihilator of x.

The support of M, $\operatorname{Supp} M$, is

$$\operatorname{Supp} M = \{p \in \operatorname{Spec} R \mid M \otimes R_p \neq 0\}.$$

Lemma 2.2. *Let A_i, A, B and C be R-modules. If we have an exact sequence*

$$0 \to A \to B \to C \to 0$$

then $\operatorname{Ass} A \subset \operatorname{Ass} B \subset \operatorname{Ass} A \cup \operatorname{Ass} C$ *and* $\operatorname{Supp} B = \operatorname{Supp} A \cup \operatorname{Supp} C$.

Also $\operatorname{Supp} \bigoplus_i A_i = \bigcup_i \operatorname{Supp} A_i$.

Proof. [6, IV.1, Proposition 3] and [6, II.4, Proposition 16]. □

Lemma 2.3. *Let B and C be R-modules. If we have an exact sequence*

$$B \xrightarrow{f} C \to 0$$

then $\operatorname{Ass} C \subset \overline{\operatorname{Ass} B}$.

Proof. Follows from [6, II.4, Proposition 16] and [6, IV.1, Proposition 7] □

Lemma 2.4. *If M is an R-module, then* $\operatorname{Supp} M = \overline{\operatorname{Ass} M}$.

Proof. [6, IV.1, Proposition 7]. □

Lemma 2.5. *Suppose (R, m) is a local ring. If M is a non-trivial finitely generated R-module then there is a surjective map $M \to R/m$.*

Proof. We prove this by quotienting out all but one element of a minimal generating set and then quotienting out by m times the last generator. □

The lemma could have also been proved using Nakayama's Lemma.

2.2. *Derived categories*

The category of chain complexes of R-modules with differential of degree -1 is denoted by $C(R)$ and $D(R)$ denotes the derived category of the ring R, which is just $C(R)$ modulo weak equivalences (see [22] for more details). We make $D(R)$ into a triangulated category in the standard way. For $M \in D(R)$ or $C(R)$, $H_i(M)$ denotes the i-th homology of M. Given $A \in C(R)$, $s^i A$ is the same complex shifted up by i, $s^i A_j = A_{j-i}$ and $ds^i a = (-1)^i s^i da$. Since $s^i(\_)$ preserves weak equivalences $s^i(\_)$ extends to $D(R)$. By convention we write $s^1 = s$. $H_i(sM) = H_{i-1}(M)$. If $f : A \to B$ is a map in $D(R)$, we get a distinguished triangle,

$$A \to B \to C \to sA.$$

Where C is isomorphic to the cone on f, $\operatorname{Cone}(f)$. Applying H_* to such a triangle we get a long exact sequence

$$H_i A \to H_i B \to H_i C \to H_{i-1} A.$$

If M is an R-module we consider it as the object M in $C(R)$ or $D(R)$ with

$$M_i = \begin{cases} M & i = 0, \\ 0 & \text{else} \end{cases}$$

and trivial differential.

For a category $\mathcal{C}$ and $A, B \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(A, B)$ denotes the set of maps from A to B. We will omit the subscript $\mathcal{C}$ if it is clear which category we are working in, usually it will be $D(R)$.

There are two important full triangulated subcategories of $D(R)$ that we will use:

- $D_{\mathrm{perf}}(R)$ is the subcategory of $D(R)$ consisting of objects represented by bounded chain complexes of finitely generated projective modules.
- $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ is the subcategory of $D(R)$ consisting of objects whose homology groups are finitely generated and bounded above and below.

Lemma 2.6. *Suppose $M, N \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$, then the natural map*

$$\mathrm{Hom}_{D(R)}(M, N) \otimes R_p \to \mathrm{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p)$$

is an isomorphism of R_p modules.

Proof. Represent N by $B \in C(R)$ that is bounded above. Since $H_i(M)$ is finitely generated and bounded below we can represent M by $A \in C(R)$ that is bounded below, projective in each degree and has finitely many generators in each degree.

Let $\mathcal{H}om^i_{gr\,R\text{-}mod}(A, B) = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{R\text{-}mod}(A_{n+i}, B_n)$, be the R-module of graded R-module maps from A to B lowering degree by i. Note that for each i this product is finite since A is bounded below and B is bounded above. We put a differential on $\mathcal{H}om_{gr\,R\text{-}mod}(A, B) = \bigoplus_{i\in\mathbb{Z}} \mathcal{H}om^i_{gr\,R\text{-}mod}(A, B)$ by setting $df(x) = d(f(x)) - (-1)^i f(dx)$.

Then the natural map

$$\theta : \mathcal{H}om^i_{gr\,R\text{-}mod}(A, B) \otimes R_p \to \mathcal{H}om^i_{gr\,R_p\text{-}mod}(A \otimes R_p, B \otimes R_p)$$

is an isomorphism of R_p modules for each i since by [10, Proposition 2.10] it is an isomorphism restricted to each factor of the product, since each A_n is finitely generated, and since also the product is finite and hence commutes with the tensor product. The map θ also commutes with the differential. Seeing chain maps as 0-cycles and chain homotopies as 0-boundaries, since A is projective and bounded below, we get that $\mathrm{Hom}_{D(R)}(M, N) = H^0(\mathcal{H}om_{gr\,R\text{-}mod}(A, B))$, and similarly for $\mathrm{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p)$. Since R_p is flat by [10, Proposition 2.5] for any $L \in C(R)$, $H_i(L) \otimes R_p \to H_i(L \otimes R_p)$ is an isomorphism of R_p modules. So

$$\begin{aligned}
\mathrm{Hom}_{D(R)}(M, N) \otimes R_p &= H^0\big(\mathcal{H}om_{gr\,R\text{-}mod}(A, B)\big) \otimes R_p \\
&\xrightarrow{\cong} H^0\big(\mathcal{H}om_{gr\,R\text{-}mod}(A, B) \otimes R_p\big) \\
&\xrightarrow{\cong} H^0\big(\mathcal{H}om_{gr\,R_p\text{-}mod}(A \otimes R_p, B \otimes R_p)\big) \\
&= \mathrm{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p),
\end{aligned}$$

and so the lemma follows. □

Lemma 2.7. *Suppose $M, N \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$. Suppose $p \in \mathrm{Supp}\, H_n(M)$ and $p \notin \mathrm{Supp}\, H_i(M)$ for $i < n$. If also $p \in \mathrm{Ass}\, H_n(N)$, and $p \notin \mathrm{Supp}\, H_i(N)$ for $i > n$, then there exists a map $f : M \to N$, such that $H_n(f) \neq 0$. In particular $\mathrm{Hom}(M, N) \neq 0$.*

Proof. Since $M, N \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$ and R_p is flat, by Lemma 2.6 it is enough to show that there exists $\phi \in \mathrm{Hom}_{D(R_p)}(M \otimes R_p, N \otimes R_p)$ such that $H_n(\phi) \neq 0$. Since $p \notin \mathrm{Supp}\, H_i(M)$ for $i < n$, $H_i(M \otimes R_p) = 0$ for $i < n$ and since $p \in \mathrm{Supp}\, H_n(M)$, $H_n(M \otimes R_p) \neq 0$. So Lemma 2.5 implies that there is a map $f : M \otimes R_p \to s^n R/p \otimes R_p$ that induces a surjection on H_n. Also since $p \in \mathrm{Ass}\, H_n(N)$ and formation of associated primes commutes with localization [10, Theorem 3.1(c)] we get that $p \in \mathrm{Ass}\, H_n(N \otimes R_p)$ and thus there is an injection $g' : R/p \otimes R_p \to H_n(N \otimes R_p)$. Since $H_i(N) \otimes R_p = 0$ for $i > n$, there is a map $g : s^n R/p \otimes R_p \to N \otimes R_p$ which induces g' in H_n. The composition $gf : M \otimes R_p \to N \otimes R_p$ is non-trivial on H_n, and we are done. □

Observe that the lemma does need the finiteness condition, since $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}/p) = 0$ in $D(\mathbb{Z})$. In particular in this case the part of the proof that relies on Lemma 2.5 does not hold. A corollary of the last lemma is:

Corollary 2.8. *If $M \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$ such that $H_n(M) \otimes R_p \neq 0$ and $H_i(M) \otimes R_p = 0$ if $i < n$ then there is a map $\phi : M \to s^n R/p$ such that $H_n(\phi) \neq 0$. In particular $H_n(\phi) \otimes R_p$ is surjective.*

Proof. By Lemma 2.7 there is a map $\phi : M \to s^n R/p$ such that $H_n(\phi) \neq 0$. This implies that $H_n(\phi) \otimes R_p$ is surjective. □

2.3. Aisles and t-structures

Let $\mathbf{T}$ be a triangulated category. We give the following definitions due to Keller and Vossieck [15].

Definition 2.9. A non-empty full subcategory $\mathcal{A}$ of $\mathbf{T}$ is a *pre-aisle* if:

1) for every $X \in \mathcal{A}$, $sX \in \mathcal{A}$,
2) for every distinguished triangle $X \to Y \to Z \to sX$, if $X, Z \in \mathcal{A}$ then $Y \in \mathcal{A}$.

A pre-aisle $\mathcal{A}$ is called *cocomplete* if $\mathcal{A}$ is closed under coproducts, in other words the coproduct of any set of elements in $\mathcal{A}$ is also in $\mathcal{A}$. A pre-aisle $\mathcal{A}$ such that the inclusion $\mathcal{A} \subset \mathbf{T}$ admits a right adjoint is called an *aisle*.

Proposition 2.10. *Suppose $\mathcal{A}$ is a pre-aisle then:*

1) $0 \in \mathcal{A}$.
2) *If $X \in \mathcal{A}$ and Y is isomorphic to X then $Y \in \mathcal{A}$.*

Proof. Follows directly from the definition. □

A t-structure (see [3] or [14, Chapter X]) consists of two subcategories $\mathcal{A}, \mathcal{A}' \subset \mathbf{T}$ such that:

1) $s\mathcal{A} \subset \mathcal{A}$, $\mathcal{A}' \subset s\mathcal{A}'$.
2) For every $A \in \mathcal{A}$, $B \in s^{-1}\mathcal{A}'$, $\mathrm{Hom}(A, B) = 0$.
3) For every $X \in \mathbf{T}$ there is a triangle

$$A \to X \to B \to sA$$

with $A \in \mathcal{A}$, $B \in s^{-1}\mathcal{A}'$.

It follows easily that the decomposition in 3) is unique.

Keller and Vossieck [15] observed that a t-structure corresponds to an aisle. For a subcategory $\mathcal{A} \subset \mathbf{T}$ we let

$$\mathcal{A}^{\perp} = \{x \in \mathbf{T} \mid \mathrm{Hom}(y, x) = 0 \ \forall y \in \mathcal{A}\}.$$

Theorem 2.11. *(See [15].) A pre-aisle $\mathcal{A}$ is an aisle, that is the inclusion $\mathcal{A} \subset \mathbf{T}$ admits a right adjoint, if and only if $(\mathcal{A}, s\mathcal{A}^{\perp})$ is a t-structure.*

We will mainly consider aisles for the rest of the paper since they are equivalent to t-structures.

Associated to the t-structure $(\mathcal{A}, s\mathcal{A}^{\perp})$ there are the right adjoint of the inclusion $\mathcal{A} \subset \mathbf{T}$, $(\_)\langle\mathcal{A}\rangle : \mathbf{T} \to \mathcal{A}$ and the left adjoint of the inclusion $\mathcal{A}^{\perp} \subset \mathbf{T}$, $P_{\mathcal{A}} : \mathbf{T} \to \mathcal{A}^{\perp}$. These adjoints are called truncation functors and often denoted something like $\tau^{\leqslant 0}$ and $\tau^{\geqslant 1}$. That notation is well suited when the associated cohomological functors are involved, however we are more concerned with the aisles themselves and hence do not need the superscripts. Thus we use the topologically motivated notation.

For any $X \in \mathbf{T}$ we get a natural triangle,

$$X\langle\mathcal{A}\rangle \to X \xrightarrow{\eta_X} P_{\mathcal{A}}(X) \to sX\langle\mathcal{A}\rangle. \tag{2.12}$$

2.4. *Nullity classes*

Cocomplete pre-aisles in the triangulated category of spectra, and similar subcategories of spaces, have also been referred to as nullity classes and Bousfield classes. We will also use the term nullity class to refer to the intersection of a cocomplete pre-aisle with a full subcategory.

Definition 2.13. Let $\mathcal{D} \subset D(R)$ be a full triangulated subcategory. A *nullity class* in $\mathcal{D}$ is a full subcategory of the form $\mathcal{A} \cap \mathcal{D}$ where $\mathcal{A}$ is a cocomplete pre-aisle in $D(R)$. We let {nullity classes} denote the set of nullity classes when $\mathcal{D} = D^{\mathrm{b}}_{\mathrm{fg}}(R)$, which we order by inclusion.

Observe that using Proposition 2.10 (and Theorem 2.14 below) the nullity classes do indeed form a set since there is a set of isomorphism classes of objects in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$.

For an object $E \in D(R)$ we let $\overline{C(E)} \subset D(R)$ denote the smallest nullity class in $D(R)$ containing E. Notice that the objects in $D(R)$ with finitely generated homology form a pre-aisle but not a nullity class, so not all pre-aisles are nullity classes, however we do have the following:

Proposition 2.14. *Suppose $\mathcal{D} \subset D(R)$ is a full triangulated subcategory. Any aisle in $\mathcal{D}$ is a nullity class and any nullity class is a pre-aisle.*

Proof. Since $\mathcal{D}$ is a triangulated subcategory it is clear that any nullity class is a pre-aisle.

Suppose $\mathcal{A} \subset \mathcal{D}$ is an aisle. It is well known that $x \in \mathcal{A}$ if and only if for every $y \in \mathcal{A}^{\perp} \subset \mathcal{D}$, $\mathrm{Hom}(x, y) = 0$ (see [1, Proposition 1.1(i)] for example). This condition is closed under taking coproducts and extensions in the first variable. Of course these extensions and coproducts are in $D(R)$ and may not be in $\mathcal{D}$. Also $\mathcal{A}^{\perp} \subset s(\mathcal{A}^{\perp})$ (again see [1, Proposition 1.1(ii)] for example), and so the condition is also closed under suspension. Thus any object in $D(R)$, and hence in the full subcategory $\mathcal{D}$, that can be constructed using these operations also satisfies the condition. The fact that an aisle is a nullity class follows. □

In the proof of the last proposition, the operations can take us out of $\mathcal{D}$, but that doesn't matter since we intersect back with $\mathcal{D}$, and $\mathcal{D}$ is full.

In [1] Alonso Tarrio, Jeremias Lopez and Souto Salorio show that for any Grothendieck category $\mathcal{U}$ and any $E \in D(\mathcal{U})$, there is an associated aisle. A special case of [1, Proposition 3.2] is the following:

Theorem 2.15. *(See [1].) Let R be a commutative ring and $E \in D(R)$. If $\mathcal{A}$ is the smallest nullity class of $D(R)$ that contains E, then $\mathcal{A}$ is an aisle in $D(R)$.*

Notation. Following the topologists we denote the nullity class $\mathcal{A}$ of the proposition associated to E by $\overline{C(E)}$. In this case we will denote the functors $(\_)\langle\mathcal{A}\rangle$ by $(\_)\langle E\rangle$ and $P_{\mathcal{A}}$ by P_E. So for any $X \in D(R)$ the distinguished triangles of Eq. (2.12) become,

$$X\langle E\rangle \to X \xrightarrow{\eta_X} P_E X \to sX\langle E\rangle. \tag{2.16}$$

where $X\langle E\rangle \in \overline{C(E)}$ and $P_E X \in \overline{C(E)}^{\perp}$.

Lemma 2.17. *Suppose $E \in D(R)$. If $p \notin \operatorname{Supp} H_i(E)$ for $i \leqslant k$ then for every $X \in \overline{C(E)}$, $p \notin \operatorname{Supp} H_i(X)$ for $i \leqslant k$.*

Proof. The condition $p \notin \operatorname{Supp} H_i(E)$ for $i \leqslant k$ is clearly closed under suspension and is closed under direct sums and extensions by Lemma 2.2. So the lemma follows directly from the Definitions 2.9 and 2.13. □

3. Properties of nullity classes and the nullification functor P_E

3.1. Properties of nullity classes

Recall that for $E \in \mathcal{D} \subset D(R)$, $\overline{C(E)}$ denotes the smallest nullity class containing E. The following two easy facts will be used often enough that it is good to keep them in mind.

Lemma 3.1. *For $E, F \in D(R)$.*

1) *$E \in \overline{C(E)}$.*
2) *$F \in \overline{C(E)}$ if and only if $\overline{C(F)} \subset \overline{C(E)}$.*

Proof. These are clearly true. □

Lemma 3.2. *For $E \in D(R)$, $\overline{C(E)}$ is closed under retracts.*

Proof. This follows from the well-known Eilenberg swindle. If $X = A \oplus B$ we can consider the countable coproduct of X with itself in two different ways. $\bigoplus_{i\in\omega} X = A \oplus B \oplus A \oplus B \ldots$ or $\bigoplus_{i\in\omega} X = B \oplus A \oplus B \oplus A \ldots$. We can include the second into the first missing the first A to get a distinguished triangle,

$$\bigoplus_{i\in\omega} X \to \bigoplus_{i\in\omega} X \to A \to s\bigoplus_{i\in\omega} X.$$

So if X is in $\overline{C(E)}$, since $\overline{C(E)}$ is cocomplete, so is $\bigoplus_{i\in\omega} X$ and then Definition 2.9 2) implies that $A \in \overline{C(E)}$. □

We can put a partial order on $D(R)$ by letting $E < F \iff P_E(F) = 0$. The following shows that $<$ is indeed a partial order.

Proposition 3.3. *For $E, F \in D(R)$,*

$$E < F \quad \iff \quad \overline{C(F)} \subset \overline{C(E)}.$$

In particular for any full triangulated subcategory $\mathcal{D} \subset D(R)$, the map $\mathcal{D} \to \{\text{nullity classes}\}$, $E \mapsto \overline{C(E)}$ is order reversing. Furthermore $\overline{C(E)} = \{X \mid P_E X = 0\}$.

Proof. By definition $E < F$ if and only if $P_E F = 0$. Next looking at the triangle of Eq. (2.16), we see that $P_E F = 0$ if and only if $F\langle E\rangle = F$. If $F = F\langle E\rangle$ then $F \in \overline{C(E)}$ and so $\overline{C(F)} \subset \overline{C(E)}$. If $\overline{C(F)} \subset \overline{C(E)}$ then $F \in \overline{C(E)}$ and also $F \to F \to 0$ is a triangle so we must have that $F\langle E\rangle = F$. This proves that $E < F$ if and only if $\overline{C(F)} \subset \overline{C(E)}$. The rest of the proposition follows easily. □

3.2. Some properties of P_E

As is usual in the topological setting, we call a map f in $D(R)$ a *P_E equivalence* if $P_E(f)$ is an isomorphism in $D(R)$ and an object $A \in D(R)$, *P_E local* if $P_E(A) = A$. The following proposition is standard in the topological settings and also has analogues for any t-structure on any triangulated category.

Proposition 3.4. *Working in $D(R)$ with objects E, F,*

1) *$\eta_A : A \to P_E A$ is a P_E equivalence.*
2) *$f : A \to B$ is a P_E equivalence if and only if for all $L \in \overline{C(E)}^\perp$*

$$\operatorname{Hom}(f, L) : \operatorname{Hom}_{D(R)}(B, L) \to \operatorname{Hom}_{D(R)}(A, L)$$

   is an isomorphism.
3) *$P_E A$ is P_E local.*
4) *P_E is left adjoint to the inclusion $\overline{C(E)}^\perp \subset D(R)$.*
5) *A is P_E local if and only if $\operatorname{Hom}(s^i E, A) = 0$ for all $i \geqslant 0$ if and only if $\operatorname{Hom}(X, A) = 0$ for all $X \in \overline{C(E)}$.*
6) *Suppose $E < F$ then P_E local objects are P_F local and P_F equivalences are P_E equivalences.*

Proof. 4): For $A \in D(R)$ and $B \in \overline{C(E)}^\perp$, define $\theta : \operatorname{Hom}_{\overline{C(E)}^\perp}(P_E A, B) \to \operatorname{Hom}_{D(R)}(A, B)$ by $\theta(f) = f\eta_A$. Consider the triangle in $D(R)$,

$$A\langle E\rangle \to A \xrightarrow{\eta_A} P_E A \to sA\langle E\rangle.$$

Since $B \in \overline{C(E)}^\perp$, $\operatorname{Hom}(A\langle E\rangle, B) = 0$, which implies that θ is surjective since any map $g : A \to B$ extends over $P_E A$. Also $\operatorname{Hom}(sA\langle E\rangle, B) = 0$ which implies that the extension is unique. In other words θ is injective. Since θ is clearly natural, this completes the proof that P_E is left adjoint to the inclusion.

1): Follows since the left adjoint of an inclusion is always idempotent.

2): Consider the map

$$\mathrm{Hom}(f,\_):\mathrm{Hom}_{D(R)}(B,\_)\to\mathrm{Hom}_{D(R)}(A,\_)$$

of representable functors $D(R)\to$ *Sets*. Restricting to $\overline{C(E)}^{\perp}$ and using the adjoint relation of Part 4), the map becomes

$$\mathrm{Hom}(P_E f,\_):\mathrm{Hom}_{\overline{C(E)}^{\perp}}(P_E B,\_)\to\mathrm{Hom}_{\overline{C(E)}^{\perp}}(P_E A,\_)$$

by Yoneda's Lemma this is an isomorphism if and only if $P_E f$ is an isomorphism. In other words $Hom(P_E(f),L)$ is an isomorphism for every $L\in\overline{C(E)}^{\perp}$ if and only if f is a P_E equivalence.

3): Part 1) implies that $P_E A\to P_E P_E A$ is an equivalence and thus $P_E A$ is P_E local.

5): Since P_E is left adjoint to the inclusion, $P_E A=A\Leftrightarrow A\in\overline{C(E)}^{\perp}$, from which the result follows. This is also proved in [1, Lemma 3.1].

6): Since $E<F$, by Proposition 3.3, $\overline{C(F)}\subset\overline{C(E)}$, and so $\overline{C(E)}^{\perp}\subset\overline{C(F)}^{\perp}$. It then follows directly from Part 2) that all P_F equivalences are P_E equivalences, and directly from Part 5) that P_E local objects are P_F local. □

The functor P_E can be characterized by its universal properties. In particular the following corollary says $P_E A$ is the unique P_E local object P_E equivalent to A.

Corollary 3.5. *If $f:A\to B$ is a P_E equivalence and B is P_E local then there is an isomorphism $\phi:B\to P_E A$ such that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\eta_A"'] & B \arrow[d, "\phi"] \\ & P_E A \end{tikzcd}$$

commutes.

Proof. The map ϕ comes from Proposition 3.4(2), and its inverse from 3.4(4). The are compositions are equal to the identity come from Proposition 3.4 (2) and (4) by the uniqueness part of using universal properties as usual. □

In $D(R)$ generally direct limits do not exist. Countable homotopy direct limits in any triangulated category were constructed in [5], and any homotopy direct limits of objects in model categories are well known to exist. In the special case of $C(R)$ a construction is given in [1]. Even though direct limits in $C(R)$ are homotopy invariant (this follows since direct limits commute with homology), the direct limits cannot generally be extended to direct limits in $D(R)$; phantom maps provide a first obstruction. For these reasons when we do constructions involving direct limits we will work in $C(R)$.

Proposition 3.6.

1) *A direct limit of P_E equivalences is a P_E equivalence. In particular given a direct system $\{A(\alpha)\}_{\alpha<\lambda}$ of objects in $C(R)$, if $P_E A(1)\to P_E A(\alpha)$ is a homology equivalence for each $\alpha<\lambda$ then $P_E A(1)\to P_E colim_{\alpha<\lambda}A(\alpha)$ is a homology equivalence.*

2) *If $A \in \overline{C(E)}$ and $A \to B \xrightarrow{i} C \to sA$ is a distinguished triangle then $B \to C$ is a P_E equivalence.*

Proof. 2): Suppose $L \in \overline{C(E)}^{\perp}$, since $A \in \overline{C(E)}$, $\mathrm{Hom}_{D(R)}(A, L) = 0$ and so $\mathrm{Hom}(i, L) : \mathrm{Hom}(C, L) \to \mathrm{Hom}(B, L)$ is surjective. Also $s\overline{C(E)} \subset \overline{C(E)}$ implies that $sA \in \overline{C(E)}$ and so $\mathrm{Hom}(sA, L) = 0$. It follows that $\mathrm{Hom}(i, L)$ is injective. Thus $\mathrm{Hom}(i, L)$ is an isomorphism for all $L \in \overline{C(E)}^{\perp}$ and so by Proposition 3.4(2), $i : B \to C$ is a P_E equivalence.

1): In the proof of this part we start by working in $C(R)$. Consider the direct limit:

$$A(\lambda) = colim_{\alpha<\lambda} A(\alpha).$$

Since P_E is a functor on $C(R)$ we get a commuting diagram,

$$\begin{tikzcd}
A(1) \arrow[r, "\eta_{A(1)}"] \arrow[d] & P_E(A(1)) \arrow[d, "i"] \\
A(\lambda) = colim_{\alpha<\lambda} A(\alpha) \arrow[r] \arrow[dr, "\eta_{A(\lambda)}"'] & colim_{\alpha<\lambda} P_E A(\alpha) \arrow[d] \\
 & P_E A(\lambda).
\end{tikzcd}$$

The map i is a homology equivalence since it is a direct limit of homology equivalences. This also implies it is a P_E equivalence. Also $\eta_{A(1)}$ and $\eta_{A(\lambda)}$ are P_E equivalences by Proposition 3.4(1). Clearly the composition $i\eta_{A(1)}$ is also a P_E equivalence. So taking P_E of the above diagram and moving the vertices a bit we get a commutative diagram in $D(R)$,

$$\begin{tikzcd}
P_E A(1) \arrow[r, "a"] \arrow[d, "b"'] & P_E colim_{\alpha<\lambda} P_E A(\alpha) \arrow[d, "d"] \\
P_E A(\lambda) \arrow[r, "e"'] \arrow[ur, "c"] & P_E P_E A(\lambda)
\end{tikzcd}$$

in which a and e are isomorphisms. We will show that $a^{-1}c$ is the inverse of b.

$$a^{-1}cb = a^{-1}a = id_{P_E A(1)}$$

and

$$eba^{-1}c = daa^{-1}c = dc = e.$$

Since e is an isomorphism, this implies that $ba^{-1}c = id_{P_E A(\lambda)}$. Thus $a^{-1}c$ is the inverse of b and so $b : P_E A(1) \to P_E A(\lambda)$ is an isomorphism in $D(R)$ as required. $\square$

4. An invariant

We will let {perversity functions} denote the set of perversity functions, for us that is the increasing functions from $\mathbb{Z}$ to subsets of Spec R closed under specialization (see 2.1). Recall that {nullity classes} denotes the set of nullity classes in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ (Definition 2.13).

In this section we define maps

$$N : \{\text{perversity functions}\} \to \{\text{nullity classes}\}$$

and

$$\phi : \{\text{nullity classes}\} \to \{\text{perversity functions}\}.$$

4.1. Concise summary of the main results in the paper

We will see in this paper that the set of perversities is in bijection with the set of nullity classes in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ (Theorem 6.12). Also the set of aisles inject into the set of perversities (Theorem 5.6) and have image in the comonotone perversities (Theorem 7.9). We conjecture that they in fact are in bijection with the comonotone perversities (Conjecture 7.10).

4.2. Perversity functions

Let

$$\{\text{perversity functions}\} = \bigl\{ f : \mathbb{Z} \to \mathcal{P}(\operatorname{Spec} R) \bigm| f(n) = \overline{f(n)} \text{ and } f(n) \subset f(n+1) \bigr\},$$

where $\mathcal{P}$ is the power set. We call an element of {perversity functions} a *perversity function* or just a *perversity*. We put an order on {perversity functions} by inclusion, more precisely $f \leqslant g$ when for every n, $f(n) \subset g(n)$.

A perversity $f \in$ {perversity functions} is called *comonotone* if whenever $p' \in f(n)$ and p is maximal under p' (in other words $p \subset p'$, $p \neq p'$ and there is no q strictly between p and p'), then $p \in f(n+1)$. So for us a comonotone perversity must increase fast enough.

As pointed out to me by Halvard Faulk, our perversity functions contain the same information as traditional monotone perversity functions with the notions of comonotone often coinciding as well. The condition of closure under specialization corresponds to being monotone.

Let $\mathcal{G}$ be the set of (traditional perversity) functions,

$$\mathcal{G} = \bigl\{ g : \operatorname{Spec}(R) \to \mathbb{Z} \cup \{\infty, -\infty\} \bigr\}.$$

An element of $\mathcal{G}$ is called *monotone* if whenever $p \subset q$ then $g(p) \leqslant g(q)$, and *comonotone* if $\tilde{g}(p) = -\dim(R/p) - g(p)$ is monotone.

For any perversity function $f \in$ {perversity functions} or more generally for any function $f : \mathbb{Z} \to \mathcal{P}(\operatorname{Spec} R)$, we can associate $\theta(f) \in \mathcal{G}$ as follows:

$$\theta(f)(p) = -\min\bigl\{ f(n) \bigm| p \in f(n) \bigr\}.$$

The next proposition shows that perversities in our sense correspond to traditional monotone perversities, and that in many cases the two definitions of comonotone agree.

Proposition 4.1. *The map $\theta : \{\text{perversity functions}\} \to \mathcal{G}$ is injective and its image is the set of monotone functions.*

If $\dim(R)$ is finite, then the image of the comonotone perversities are both monotone and comonotone. If in addition for all primes p and q with p maximal under q, $\dim R/p = \dim R/q + 1$, then the image of the comonotone perversities is exactly the functions that are both monotone and comonotone.

Proof. The inverse to θ is given by θ', which for $g \in \mathcal{G}$ is defined by

$$\theta'(g)(n) = \bigl\{p \bigm| -n \leqslant g(p)\bigr\}.$$

Given any function $f : \mathbb{Z} \to \mathcal{P}(\operatorname{Spec} R)$ such that $f(n) \subset f(n+1)$, we can calculate that $f(n) \subset \theta'\theta(f)(n)$ always and that $\theta'\theta(f)(n) \subset f(n)$ since f is an increasing function.

The map θ takes perversity functions, in particular functions whose images are closed under specialization, to monotone functions. For primes q and p with $p \subset q$, let k be the height of q in R/p. When $\dim R$ is finite, we have that $\dim R/p - \dim R/q \geqslant k$. This implies that θ takes comonotone perversity functions to comonotone functions. In the other direction θ' takes monotone functions to perversity functions.

Now suppose $\dim R$ is finite and for every pair of primes $p \subset q$ with p maximal under q, $\dim R/p = \dim R/q + 1$. Then it is straightforward to check that θ' takes comonotone monotone functions to comonotone perversity functions. From the observed relations the proposition follows. $\square$

4.3. Definition of N

For a perversity function f let $M(f) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in f(n)} s^n R/p$ and $N(f) = \overline{C(M(f))} \cap D^{\mathrm{b}}_{\mathrm{fg}}(R)$ be the associated nullity class.

4.4. Definition of ϕ

Let $\mathcal{A} \subset D^{\mathrm{b}}_{\mathrm{fg}}(R)$ be a nullity class. Define a perversity function $\phi(\mathcal{A})$ by letting $p \in \phi(\mathcal{A})(n)$ if there is $M \in \mathcal{A}$ such that $p \in \operatorname{Supp} H_n(M)$. So

$$\phi(\mathcal{A})(n) = \bigl\{p \in \operatorname{Spec} R \bigm| \exists M \in \mathcal{A} \text{ with } p \in \operatorname{Supp} H_n(M)\bigr\}.$$

Note that $\phi(\mathcal{A}) \in \{\text{perversity functions}\}$ since if $M \in \mathcal{A}$ and $p \in \operatorname{Supp} H_n(M)$ then $sM \in \mathcal{A}$ and $p \in \operatorname{Supp} H_{n+1}(sM)$, so $\phi(\mathcal{A})$ is increasing, also each $\phi(\mathcal{A})(n)$ is clearly closed under specialization from the way they are defined. Under the correspondence of Hopkins and Neeman [17] the $\phi(\mathcal{A})(n)$ correspond to the thick subcategories of $D_{\mathrm{perf}}(R)$ generated by the usual truncations of $\mathcal{A}$ by dimension.

As advertised in the abstract ϕ can be considered an invariant of t-structures in $D(R)$ by simply intersecting the associated aisle in $D(R)$ with $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ and taking ϕ as defined. Alternatively we could just define ϕ exactly as above.

Lemma 4.2. *N and ϕ are order preserving.*

Proof. That ϕ is order preserving follows immediately from the definition. For perversity functions f, g, if $f \leqslant g$ then $M(f)$ is a retract of $M(g)$ so $M(f) \in \overline{C(M(g))}$ by Lemma 3.2. Therefore $\overline{C(M(f))} \subset \overline{C(M(g))}$ by Lemma 3.1 and N is seen to be order preserving. $\square$

5. Complete invariant

In this section we give a short proof that when restricted to aisles in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, ϕ is injective. In other words ϕ is a complete invariant of t-structures in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$.

Lemma 5.1. *For any finitely generated R-module B, $\bigoplus_{p \in \operatorname{Supp} B} R/p < B$.*

Proof. Since B is finitely generated by [6, IV.1, Theorem 1] or [10, Proposition 3.7] it has a decomposition

$$0 = B_0 \subset B_1 \subset \cdots \subset B_n = B$$

such that $B_i/B_{i-1} = R/p(i)$ for some primes $p(i)$. By Lemmas 2.2, 2.3 and 2.4 each $p(i) \in \operatorname{Supp} B$. Thus Definition 2.9(2), Lemma 3.2 and Proposition 3.3 then imply that $\bigoplus_{p \in \operatorname{Ass} B} R/p < B$ and so also $\bigoplus_{p \in \operatorname{Supp} B} R/p < B$. $\square$

Lemma 5.2. *For any R-module B, $\bigoplus_{p \in \operatorname{Supp} B} R/p < B$.*

Proof. Note that since we will use Proposition 3.6 we work in $C(R)$. Let $\{x_i\}_{i<\lambda} \subset B$ be a generating set. For $\alpha \leqslant \lambda$, let $B(\alpha) \subset B$ be the submodule generated by $\{x_i\}_{i<\alpha}$. Then $B = B(\lambda)$ and we will prove the lemma by induction.

Assume $\bigoplus_{p \in \operatorname{Supp} B} R/p < B(\gamma)$ for all $\gamma < \alpha$. In other words $0 \to B(\gamma)$ is a $P_{\bigoplus_{p \in \operatorname{Supp} B} R/p}$ equivalence. If α is a limit ordinal then $B(\alpha) = colim_{\gamma<\alpha} B(\alpha)$ so $0 \to B(\alpha)$ is a $P_{\bigoplus_{p \in \operatorname{Supp} B} R/p}$ equivalence by Proposition 3.6. Thus $\bigoplus_{p \in \operatorname{Supp} B} R/p < B(\alpha)$.

If $\alpha = \gamma + 1$ then consider the exact sequence

$$0 \to B(\gamma) \to B(\alpha) \to M = B(\alpha)/B(\gamma) \to 0.$$

The image of x_α generates M and, from Lemma 2.2, $\operatorname{Supp} M \subset \operatorname{Supp} B(\alpha) \subset \operatorname{Supp} B$. Thus from Lemma 5.1, $\bigoplus_{p \in \operatorname{Supp} B} R/p < M$. Using Proposition 3.3, Definition 2.9(2) and the induction hypothesis, we see that $\bigoplus_{p \in \operatorname{Supp} B} R/p < B(\alpha)$. The lemma now follows by induction. $\square$

Lemma 5.3. *For every $M \in D(R)$ with homology in only finitely many degrees,*

$$\bigoplus_{i \in \mathbb{Z}} \bigoplus_{p \in \operatorname{Supp} H_i(M)} s^i R/p < M.$$

Proof. By finiteness M has a decomposition

$$0 \to M_r \to M_{r+1} \to \cdots \to M_s = M$$

such that for every i, $M_i \to M_{i+1} \to s^{i+1}H_{i+1}(M) \to sM_i$ is a distinguished triangle. Thus it follows easily from Lemma 5.2 and Definition 2.9(2), that $\bigoplus_{i\in\mathbb{Z}} \bigoplus_{p\in \operatorname{Supp} H_i(M)} s^i R/p < M$. $\square$

A similar proof of the last lemma works in the category of bounded above complexes and presumably the lemma can be proved in the full derived category using an idea like that in Lemma 5.2. Next we give a technical lemma.

Lemma 5.4. *Let $\mathcal{D} \subset D(R)$ be any full triangulated subcategory. Let $\mathcal{A}$ be an aisle in $\mathcal{D}$ and $M \in \mathcal{D}$. If $p \in \operatorname{Ass} H_n P_{\mathcal{A}}(M)$ then $p \in \operatorname{Supp} H_n(M)$ or there exists $N \in \mathcal{A}$, $p \in \operatorname{Ass} H_{n-1}(N)$.*

Proof. There is a distinguished triangle

$$M\langle\mathcal{A}\rangle \xrightarrow{f} M \to P_{\mathcal{A}}M \to sM\langle\mathcal{A}\rangle.$$

From this we get a short exact sequence

$$0 \to H_n(M)/\operatorname{im} H_n(f) \to H_n(P_{\mathcal{A}}M) \to \ker H_{n-1}(f) \to 0.$$

By Lemmas 2.3 and 2.4, $\operatorname{Ass} H_n(M)/\operatorname{im} H_n(f) \subset \operatorname{Supp} H_n(M)$ and also $\operatorname{Ass}\ker H_{n-1}(f) \subset \operatorname{Ass} H_{n-1}M\langle\mathcal{A}\rangle$. So by Lemma 2.2, $\operatorname{Ass} H_n P_{\mathcal{A}}M \subset \operatorname{Supp} H_n M \cup \operatorname{Ass} H_{n-1} M\langle\mathcal{A}\rangle$. Since $M\langle\mathcal{A}\rangle \in \mathcal{A}$ the lemma follows. $\square$

Proposition 5.5. *Let $\mathcal{D} = D^{\mathrm{b}}_{\mathrm{fg}}(R)$ or $D_{\mathrm{perf}}(R)$. Suppose $\mathcal{A}$ is an aisle in $\mathcal{D}$ and $M \in \mathcal{D}$. Suppose for every n and for every $p \in \operatorname{Ass} H_n(M)$, there exists $l \leqslant n$, $N \in \mathcal{A}$ and $p \in \operatorname{Supp} H_l(N)$, then $M \in \mathcal{A}$.*

Proof. Assume $P_{\mathcal{A}}M \neq 0$. Let n be the largest such that $H_n(P_{\mathcal{A}}M) \neq 0$. This n exists since $P_{\mathcal{A}}M \in \mathcal{D}$. Then there exists a prime p such that $p \in \operatorname{Ass} H_n(P_{\mathcal{A}}M)$. Fix this n and p for the rest of the proof. Thus by Lemma 5.4, either $p \in \operatorname{Supp} H_n(M)$ or $p \in \operatorname{Ass} H_{n-1}(N)$ for some $N \in \mathcal{A}$. By the hypotheses if $p \in \operatorname{Supp} H_n(M)$ then $p \in \operatorname{Supp} H_l(N)$ for some $N \in \mathcal{A}$ and $l \leqslant n$. Since if $N \in \mathcal{A}$ then $sN \in \mathcal{A}$, we see that there exists $N \in \mathcal{A}$ such that $p \in \operatorname{Supp} H_n(N)$ and $p \notin \operatorname{Supp} H_l(N)$ for $l < n$. So Lemma 2.7 says that $\operatorname{Hom}(N, P_{\mathcal{A}}M) \neq 0$, which contradicts the fact (see [1, Proposition 1.1]) that $P_{\mathcal{A}}M \in \mathcal{A}^{\perp}$. So $P_{\mathcal{A}}M = 0$ and $M \in \mathcal{A}$. $\square$

It may look like the proof of the last proposition should extend to all nullity classes or even to any full subcategory $\mathcal{D} \subset D(R)$. Observe though that there are finiteness conditions needed in the results the proof calls on. In Section 8 we will show there is a proper class of t-structures in $D\mathbb{Z}$ so, considering the next theorem, some finiteness or other assumptions are needed.

Theorem 5.6. *Let $\mathcal{D} = D^{\mathrm{b}}_{\mathrm{fg}}(R)$ or $D_{\mathrm{perf}}(R)$. Suppose $\mathcal{A}$ is a nullity class in $\mathcal{D}$ and $\mathcal{A}'$ is an aisle in $\mathcal{D}$, then $\mathcal{A} \subset \mathcal{A}'$ if and only if for every $n \in \mathbb{Z}$, $\phi(\mathcal{A})(n) \subset \phi(\mathcal{A}')(n)$. Thus $\phi : \{\text{aisles in } \mathcal{D}\} \to \{\text{perversity functions}\}$ is injective.*

In addition for any aisle $\mathcal{A} = N(\phi(\mathcal{A}))$.

Proof. If $\mathcal{A} \subset \mathcal{A}'$ then it is clear from the definition that $\phi(\mathcal{A})(n) \subset \phi(\mathcal{A}')(n)$ for all n. Now suppose $\phi(\mathcal{A})(n) \subset \phi(\mathcal{A}')(n)$ for all n. Let $M \in \mathcal{A}$, then for every n and $p \in \operatorname{Supp} H_n(M)$,

$p \in \phi(\mathcal{A})(n) \subset \phi(\mathcal{A}')(n)$ which means there exists $N \in \mathcal{A}'$ with $p \in \operatorname{Supp} H_n(N)$. So it follows from Proposition 5.5 that $M \in \mathcal{A}'$ and so $\mathcal{A} \subset \mathcal{A}'$. Injectivity follows immediately since by Proposition 2.14 all nullity classes are aisles.

It is clear from the definitions that $\phi(\mathcal{A}) \subset \phi(N(\phi(\mathcal{A})))$ and it follows from Lemma 2.17 that $\phi(N(\phi(\mathcal{A}))) \subset \phi(\mathcal{A})$ and so $\phi(N(\phi(\mathcal{A}))) = \phi(\mathcal{A})$. Thus the first part of the theorem implies that $N(\phi(\mathcal{A})) \subset \mathcal{A}$. For any $M \in \mathcal{A}$ and $p \in \operatorname{Supp} H_i(M)$ the definitions of ϕ and N imply that $s^i R/p$ is a retract of $M(\phi(\mathcal{A}))$ and so by Lemma 3.2 $s^i R/p \in N(\phi(\mathcal{A}))$. Next Lemma 5.3 says that $M \in N(\phi(\mathcal{A}))$ and thus $\phi(\mathcal{A}) \subset N(\phi(\mathcal{A}))$. □

The following corollary follows immediately.

Corollary 5.7. *Any aisle in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ is of the form $\overline{C(E)} \cap D^{\mathrm{b}}_{\mathrm{fg}}(R)$ for some $E \in D(R)$.*

Proof. We can let $E = M(\phi(\mathcal{A}))$. □

6. Nullity classes in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$

In this section for R of finite dimension, we classify nullity classes in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$. For such R this also gives us another proof that ϕ is a complete invariant of t-structures.

We use $k(p)$ to denote $(R/p)_{(0)} = (R/p) \otimes R_p$.

Lemma 6.1. *$P_E B = 0$ implies that for every M, $P_{E\otimes M}(B \otimes M) = 0$.*

Proof. Let $\mathcal{S} = \{C \in D(R) \mid C \otimes M \in \overline{C(E \otimes M)}\}$. Since $\_ \otimes M$ preserves triangles, $\mathcal{S}$ is a pre-aisle and since $\_ \otimes M$ commutes with coproducts $\mathcal{S}$ is cocomplete. Also $E \subset \mathcal{S}$ and so $\overline{C(E)} \subset \mathcal{S}$ and thus $\overline{C(E)} \otimes M \subset \overline{C(E) \otimes M}$. Since $P_E B = 0$, $B \in \overline{C(E)}$ by Proposition 3.3. So $B \otimes M \in \overline{C(E)} \otimes M \subset \overline{C(E \otimes M)}$, and hence, again by Proposition 3.3, $P_{E\otimes M}(B \otimes M) = 0$. □

Lemma 6.2. *For $M \in D(R)$, if $H_*(M) = 0$ for $* < 0$, then $P_R M = 0$.*

Proof. Straightforward. □

Lemma 6.3. *If $P_A B = 0$ and $H_i(M) = 0$ for $i < 0$ then $P_A(B \otimes M) = 0$.*

Proof. Lemma 6.2 says that $R < M$ so Lemma 6.1 implies that $B < B \otimes M$. Since $A < B$ by assumption, the transitivity of $<$ (Proposition 3.3) implies that $A < B \otimes M$ and we are done. □

Lemma 6.4. *If $M \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$ and $q \in \operatorname{Supp} H_0(M)$, then $M < k(q)$.*

Proof. We know that $M < s^k M$ for each $k \geqslant 0$. So since $M \in D^b_{fg}(R)$, we can also assume that $q \notin \operatorname{Supp} H_i(M)$ for $i < 0$. Applying Corollary 2.8 we get a map $\phi : M \to R/p$ so that $H_0(\phi) \otimes R_p : H_0(M) \otimes R_p \to k(p)$ is surjective. Since $H_i(M) = 0$ for $i < 0$ this implies that $H_0(M \otimes k(p)) \neq 0$.

By Lemma 6.2, $R < k(q)$. So by Lemma 6.1, $M < M \otimes k(q)$. By [5, Lemma 2.17], $M \otimes k(q)$ is a direct sum of suspensions of $k(q)$. Since $H_0(M \otimes k(p)) \neq 0$, in degree 0 this direct sum is non-trivial. So the result follows from Lemma 3.2 and Proposition 3.3. □

The last lemma does not always hold for $M \in D(R)$ as we see by taking $R = \mathbb{Z}$, $M = \mathbb{Q}$ and $q = (p)$ for any non-zero prime $p \in \mathbb{Z}$.

Lemma 6.5. $\operatorname{Ass}(k(q)) = \{q\}$.

Proof. Let $\frac{x}{u} \in k(q)$ with $x \in R/q$ and $u \in R \setminus q$. Clearly $q \subset \operatorname{ann} \frac{x}{u}$ and if $l \in \operatorname{ann} \frac{x}{u}$ then $vlx = 0$ for some $v \in R \setminus q$. Since R/q is an integral domain this implies either $x = 0$ or $l \in q$. So $\operatorname{ann} \frac{x}{u} = R$ or $\operatorname{ann} \frac{x}{u} \subset q$ and we are done. $\square$

Proposition 6.6. *Suppose* $\dim R$ *is finite and* $S \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$. *For every* $p \in \operatorname{Ass} H_0(S)$ *and* q *such that* $p \subset q$, $S < s^{\dim R/q} R/q$. *In particular,* $S < s^{\dim R} R/q$.

Proof. First observe that for each q, $\dim R/q$ is finite since $\dim R$ is. Fix $p \in \operatorname{Ass} H_0(S)$. Looking at a particular q, assume for every q' with $q \subset q'$, $q' \neq q$ the lemma holds. Let M be defined to make the following sequence short exact

$$0 \to R/q \xrightarrow{f} k(q) \to M \to 0.$$

By Lemma 6.5, $\operatorname{Ass}(k(q)) = \{q\}$ and so by Lemma 2.3, $\operatorname{Ass} M \subset \operatorname{Supp}(k(q)) = \bar{q}$. The above sequence remains short exact after tensoring with the flat module R_q. Also $f \otimes R_q$ is an isomorphism and so $M \otimes R_q = 0$. So $q \notin \operatorname{Supp} M$, and $\operatorname{Supp} M \subset \bar{q} - \{q\}$. Therefore by the induction hypothesis and Lemma 5.2, $S < s^{\dim R/q-1} M$. By Definition 2.9(2) and Lemma 6.4, $S < k(q) < s^{\dim R/q} k(q)$. The short exact sequence above gives a triangle

$$s^{\dim R/q-1} M \to s^{\dim R/q} R/q \to s^{\dim R/q} k(q) \to s^{\dim R/q} M$$

so using the fact that $\overline{C(S)}$ is closed under extensions and Proposition 3.3, $S < s^{\dim R/q} R/q$. Notice that if $\dim R/q = 0$ then q is maximal and $k(q) = R/q$, so $S < k(q) = R/q$. This completes the proof of the first statement of the proposition. The second statement follows since $\dim R/q \leqslant \dim R$. $\square$

Lemma 6.7. *Suppose that* $\dim R$ *is finite and* $M \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$. *If* $p' \in \operatorname{Ass} H_n(M)$ *and* $p' \subset p$ *then there exist* $N \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$ *such that* $p \in \operatorname{Ass} H_n(N)$, $M < N$ *and for every* i, *if* $p'' \in \operatorname{Ass} H_i(N)$ *then* $p \subset p''$.

Proof. Since $M < sM$ we can assume that n is the smallest such that $p' \in \operatorname{Ass} H_n(M)$ with $p' \subset p$. Thus $H_i(M \otimes R_p) = 0$ for $i < n$. Suppose $p = (x_1, \dots, x_s)$ and let $K = K(x_1, \dots, x_s) = \bigotimes_{i=1}^{s} \operatorname{Cone}(R \xrightarrow{\cdot x_i} R)$ denote the Koszul complex. Let $N = M \otimes K$. By Lemma 6.3 $M < N$. By [10, Proposition 17.14] if $y \in p$ then y annihilates $H_*(N)$, hence for every i, if $p'' \in \operatorname{Ass} H_i(N)$ then $p \subset p''$. Since $H_i(M \otimes R_p) = 0$ for $i < n$ we calculate that

$$\begin{aligned} H_n(N) \otimes R_p &\cong H_n\big((M \otimes R_p) \otimes_{R_p} (K \otimes R_p)\big) \\ &\cong H_n(M \otimes R_p) \otimes_{R_p} H_0(K \otimes R_p) \cong \big(H_n(M) \otimes R_p\big) \otimes_{R_p} (R/p \otimes R_p). \end{aligned}$$

Since $p' \in \operatorname{Ass} H_n(M)$ with $p' \subset p$, $H_n(M) \otimes R_p \neq 0$. So using Lemma 2.5 we have a surjective map $H_n(M) \otimes R_p \to R/p \otimes R_p$. Then using the facts that $\_ \otimes (R/p \otimes R_p)$ preserves surjections and $R/p \otimes R_p \otimes_{R_p} R/p \otimes R_p \neq 0$, we see that the last module above is not 0. Hence $H_n(N) \otimes R_p \neq 0$ so we must have that $p \in \operatorname{Ass} H_n(N)$. This shows that N satisfies the desired conditions. □

Lemma 6.8. *Suppose* $\dim R$ *is finite,* $M \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$ *and* $p \in \operatorname{Spec}(R)$. *If* $p \in \operatorname{Ass} H_i(M)$ *and* $p \subset p'$ *then* $M < s^t R/p'$ *for all* $t \geqslant i$.

Proof. Note that since we always have $L < sL$, to show that $M < s^t R/p'$ for all $t \geqslant i$, it is enough to show that $M < s^i R/p'$. This fact is used a few times in the proof.

Fix a prime p and assume that the lemma is true for each $M \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$ and each prime p'' such that $p \subsetneq p''$. We wish to show the lemma is true for p. So assume that $p \in \operatorname{Ass} H_i(M)$ and let p' be any prime such that $p \subsetneq p'$. By Lemma 6.7 there is an N such that $M < N$ and $p' \in \operatorname{Ass} H_i(N)$. So by the induction hypothesis $N < s^i R/p'$, and so, since $M < N$, by transitivity $M < s^i R/p'$. Thus all that remains to prove is that $M < s^i R/p$.

From Proposition 6.6 we know that for some k, $M < s^k R/p$. Let $j \leqslant k$ be the smallest such that $M < s^j R/p$. If $j \leqslant i$ we are done, otherwise all that remains is to show that $M < s^{j-1} R/p$. Using Lemma 6.7, there exists $N \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$ such that $M < N$, $H_i(N) \otimes R_p \neq 0$, and if $p'' \in \operatorname{Ass} H_n(N)$ for some n then $p \subset p''$. Since $N < sN$, N has bounded homology and $j - 1 \geqslant i$, by suspending N if necessary we can assume that $j - 1$ is the smallest such that $H_{j-1}(N) \otimes R_p \neq 0$. So in summary $M < N$, $H_{j-1}(N) \otimes R_p \neq 0$, $H_l(N) \otimes R_p = 0$ if $l < j - 1$, and if $p'' \in \operatorname{Ass} H_n(N)$ for some n then $p \subset p''$.

Next we will show that $M < s^{j-1} R/p$. By Corollary 2.8 there is a map $\phi : N \to s^{j-1} R/p$ such that $H_{j-1}(\phi) \otimes R_p$ is surjective. Consider the long exact sequence

$$\ldots H_t(N) \xrightarrow{H_t(\phi)} H_t\big(s^{j-1} R/p\big) \to H_t(\operatorname{Cone}\phi) \to H_{t-1}(N) \to \cdots.$$

When $t \neq j - 1$, $H_t(s^{j-1} R/p) = 0$. Thus $H_t(\operatorname{Cone}\phi)$ is a submodule of $H_{t-1}(N)$, so by Lemma 2.2, $\operatorname{Supp} H_t(\operatorname{Cone}\phi) \subset \operatorname{Supp} H_{t-1}(N)$. In case $t = j - 1$, tensoring the above exact sequence with R_p gives another exact sequence. We know that $H_{j-1}(\phi) \otimes R_p$ is surjective, so $\operatorname{Supp}(H_{j-1}(s^{j-1})/\mathrm{im} H_{j-1}(\phi)) \subset \overline{p} - \{p\}$. Also $\operatorname{Supp} H_{j-1}(N) \subset \overline{p} - \{p\}$ and so, using Lemma 2.2, $\operatorname{Supp} H_{j-1}(\operatorname{Cone}\phi) \subset \overline{p} - \{p\}$. Thus the induction hypotheses says that for each t and $p' \in \operatorname{Supp} H_t(\operatorname{Cone}(\phi))$ with $p' \neq p$, $N < s^t R/p'$. Also $M < s^t R/p$ for all $t \geqslant j$. Since $M < N$ and by Proposition 3.3 $<$ is transitive, we see that $M < s^t R/p'$ for all $p' \in \operatorname{Supp}(\operatorname{Cone}(\phi))$. Thus by Lemma 5.3 we get that $M < \operatorname{Cone}(\phi)$.

Recall we have a triangle

$$N \to s^{j-1} R/p \to \operatorname{Cone}(\phi) \to sN.$$

Since $M < N$ and $M < \operatorname{Cone}(\phi)$ we get by Proposition 3.3 that $N, \operatorname{Cone}(\phi) \in \overline{C(M)}$ and so $s^{j-1} R/p \in \overline{C(M)}$. In other words, by Proposition 3.3 $M < s^{j-1} R/p$ and we are done. □

I believe there are slightly different proofs of 6.6, 6.7 and 6.8 that can get rid of the condition that $\dim R$ is finite. Recall the definitions of ϕ and N from Section 4.

Proposition 6.9. *For any nullity class $\mathcal{A} \subset D^{\mathrm{b}}_{\mathrm{fg}}(R)$, $\mathcal{A} \subset N\phi(\mathcal{A})$.*

Proof. Let $S \in \mathcal{A}$. Then by definition for every $n \in \mathbb{Z}$ and $p \in \operatorname{Supp} H_n(S)$, $p \in \phi(\mathcal{A})(n)$. Thus $M(\phi(\mathcal{A})) < \bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in \operatorname{Supp} H_n(S)} R/p$ by Lemma 3.2 since $\bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in \operatorname{Supp} H_n(S)} R/p$ is a retract of $M(\phi(\mathcal{A}))$. So Lemma 5.3 and Proposition 3.3 imply that $M(\phi(\mathcal{A})) < S$ and therefore $S \in N\phi(\mathcal{A})$, so $\mathcal{A} \subset N\phi(\mathcal{A})$. □

Proposition 6.10. *Suppose* dim R *is finite. For every nullity class $\mathcal{A} \subset D^{\mathrm{b}}_{\mathrm{fg}}(R)$, $N\phi(\mathcal{A}) \subset \mathcal{A}$.*

Proof. Suppose $p \in \phi(\mathcal{A})(n)$, then there exist $S(p,n) \in \mathcal{A}$ with $p \in \operatorname{Supp} H_n(S(p,n))$. By Lemma 6.8, $S(p,n) < s^n R/p$. Therefore $\bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in\phi(\mathcal{A})(n)} S(p,n) \in \mathcal{A}$ and

$$\bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in\phi(\mathcal{A})(n)} S(p,n) < \bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in\phi(\mathcal{A})(n)} s^n R/p = M\bigl(\phi(\mathcal{A})\bigr).$$

So $M(\phi(\mathcal{A})) \in \overline{C(\bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in\phi(\mathcal{A})(n)} S(p,n))} \subset \mathcal{A}$, and $N\phi(\mathcal{A}) \subset \mathcal{A}$ as desired. □

Again some finiteness condition is needed, since $(0) \in \operatorname{Ass} \mathbb{Q}$, but $\mathbb{Q} \not< \mathbb{Z}$ and so $\mathbb{Z} \notin \overline{C(\mathbb{Q})}$.

Proposition 6.11. *For any perversity function $f : \mathbb{Z} \to \operatorname{Spec} R$, $\phi N f = f$.*

Proof. Suppose $p \in f(n)$ then since $s^n R/p$ is a retract of $M(f)$ by Lemma 3.2 we see that $s^n R/p \in Nf$ and so since $p \in \operatorname{Ass} H_n(s^n R/p)$, $p \in (\phi N f)(n)$.

Now suppose $p \in \phi N f(n)$. Then there is a $S \in N(f) = \overline{C(M(f))}$ such that $p \in \operatorname{Supp} H_n(S)$. By the contrapositive of Lemma 2.17 we see that $p \in \operatorname{Supp} H_i(M(f))$ for some $i \leqslant n$ and thus it follows that $p \in f(i)$. Since f is increasing, $f(i) \subset f(n)$ and we see that $p \in f(n)$ as required. □

The next theorem provides a classification of nullity classes in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$.

Theorem 6.12. *Suppose* dim R *is finite. Then* $\phi : \{\text{nullity classes}\} \to \{\text{perversity functions}\}$ *and* $N : \{\text{perversity functions}\} \to \{\text{nullity classes}\}$ *are inverse bijections of partially ordered sets.*

Proof. This follows from the last three Lemmas 6.9, 6.10 and 6.11. □

The following corollary follows immediately.

Corollary 6.13. *When* dim R *is finite all nullity classes in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ are of the form $\overline{C(E)} \cap D^{\mathrm{b}}_{\mathrm{fg}}(R)$ for some $E \in D(R)$.*

As another corollary we give another proof of Theorem 5.6, at least when dim R is finite.

Corollary 6.14. *Suppose* dim R *is finite. Suppose $\mathcal{A}$, A' are aisles in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, then $\mathcal{A} \subset \mathcal{A}'$ if and only if for every $n \in \mathbb{Z}$, $\phi(\mathcal{A})(n) \subset \phi(\mathcal{A}')(n)$. Thus* $\phi : \{\text{aisles in } D^{\mathrm{b}}_{\mathrm{fg}}(R)\} \to \{\text{perversity functions}\}$ *is injective.*

Proof. Follows from Theorem 6.12. □

We can consider constant functions $f \in \{\text{perversity functions}\}$ so that $f(i) = f(j)$ for all $i, j \in \mathbb{Z}$. Taking N of such a function we get a nullity class $N(f)$ that is closed under desuspension. As such it is a thick subcategory in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, yet we do not get all thick subcategories in this way. For example consider $R = \mathbb{Z}/4$. $\operatorname{Spec}\mathbb{Z}/4 = \{(2)\}$, so a constant function $f : \mathbb{Z} \to \operatorname{Spec}\mathbb{Z}/4$ is either $f(n) = \emptyset$ or $f(n) = \{(2)\}$. If $f(n) = \emptyset$ then $N(f) = \{0\}$, the class consisting of only the trivial complex, and if $f(n) = \{(2)\}$ then $N(f) = D^{\mathrm{b}}_{\mathrm{fg}}(R)$. However $D_{\mathrm{perf}}(R) \subset D^{\mathrm{b}}_{\mathrm{fg}}(R)$ is another thick subcategory, and $D_{\mathrm{perf}}(R) \neq D^{\mathrm{b}}_{\mathrm{fg}}(R)$ since $\mathbb{Z}/2 \notin D_{\mathrm{perf}}(R)$ as it only has infinite resolutions. So in $D^{\mathrm{b}}_{\mathrm{fg}}(\mathbb{Z}/4)$ there are more thick subcategories than nullity classes closed under desuspension. Since $\bigoplus_{n\geqslant 0}$ generates $D_+(R)$, using Lemma 6.1, after allowing intersection back with $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, $\overline{C(N(f))}$ will always be a tensor ideal with respect to tensoring with $D_+(R)$. Notice that $D_{\mathrm{perf}}(\mathbb{Z}/4)$ is not such an ideal since $\mathbb{Z}/4 \in D_{\mathrm{perf}}(\mathbb{Z}/4)$ but $\mathbb{Z}/2 \cong \mathbb{Z}/4 \otimes_{\mathbb{Z}/4} \mathbb{Z}/2$ is not. On the other hand $D^{\mathrm{b}}_{\mathrm{fg}}(\mathbb{Z}/4)$ is also not a tensor ideal in all of $D(\mathbb{Z}/4)$ since $\mathbb{Z}/2 \otimes \mathbb{Z}/2 \notin D^{\mathrm{b}}_{\mathrm{fg}}(\mathbb{Z}/4)$. Nevertheless considering constant functions in $\{\text{perversity functions}\}$ simply as subsets of $\operatorname{Spec} R$ we do get the following corollary of Theorem 6.12.

Corollary 6.15. *Suppose* $\dim R$ *is finite, then* ϕ *and* N *induce order preserving bijections between the set of nullity classes in* $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ *closed under desuspension and subsets of* $\operatorname{Spec} R$ *closed under specialization.*

Proof. Follows directly from Theorem 6.12 and the definitions of ϕ and N. $\square$

It is tempting to think that by restricting to $D_{\mathrm{perf}}(R)$, we should be able to recover the result of Hopkins and Neeman [12,17], but I know of no way to do that.

7. Image of invariant

In this section we get some control over what the image of ϕ is when restricted to t-structures. The main object of the section is to show that if $\mathcal{A} \subset D^{\mathrm{b}}_{\mathrm{fg}}(R)$ is an aisle then $\phi(\mathcal{A})$ must be a comonotone perversity (see 4.2 and Theorem 7.9). This means that the perversity function $\phi(\mathcal{A}) : \mathbb{Z} \to \operatorname{Spec} R$ must increase in a very particular way. The classification of nullity classes from Section 6 is only used to make the proof of Lemma 7.1 a little easier, but could be avoided.

As a motivating example let us work in $D(\mathbb{Z}_{(p)})$ and consider $P_{\mathbb{Z}/p} s\mathbb{Z}_{(p)}$. Since we have a short exact sequence

$$0 \to \mathbb{Z}_{(p)} \xrightarrow{\times p} \mathbb{Z}_{(p)} \to \mathbb{Z}/p \to 0$$

we have a triangle

$$\mathbb{Z}/p \to s\mathbb{Z}_{(p)} \xrightarrow{\times p} s\mathbb{Z}_{(p)} \to s\mathbb{Z}/p$$

so $s\mathbb{Z}_{(p)} \overset{\times p}{\to} s\mathbb{Z}_{(p)}$ is a $P_{\mathbb{Z}/p}$ equivalence. Taking colimits we see that

$$s\mathbb{Z}_{(p)} \to colim(s\mathbb{Z}_{(p)} \xrightarrow{\times p} s\mathbb{Z}_{(p)} \xrightarrow{\times p} \cdots) = s\mathbb{Q}$$

is a $P_{\mathbb{Z}/p}$ equivalence. Also $\mathrm{Hom}(s^i\mathbb{Z}/p, s\mathbb{Q}) = 0$ for all $i \geqslant 0$, so $P_{\mathbb{Z}/p}s\mathbb{Z}_{(p)} = s\mathbb{Q}$. However $s\mathbb{Z}_{(p)} \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathbb{Z}_{(p)})$ but $s\mathbb{Q} \notin D^{\mathrm{b}}_{\mathrm{fg}}(\mathbb{Z}_{(p)})$. This implies that the nullity class $\mathcal{A} = \overline{C(\mathbb{Z}/p)}$ is not an aisle in $D^{\mathrm{b}}_{\mathrm{fg}}(\mathbb{Z}_{(p)})$. In fact we would need to add $s\mathbb{Z}_{(p)}$ to $\mathcal{A}$ to make it an aisle. It is this basic phenomenon that stops many nullity classes from being aisles.

Recall from 4.4 that for a perversity function f, $M(f) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in f(n)} s^n R/p$. Also recall that a prime p is maximal under a prime q if $p \subset q$, $p \neq q$ and if for any prime r such that $p \subset r \subset q$, $r = p$ or $r = q$.

Lemma 7.1. *Suppose R is a local ring with maximal ideal m, and p a prime maximal under m. Let $h \in m - p$ be any element. Suppose $f \in \{\text{perversity functions}\}$ such that $m \in f(n)$. If $N = s^n(R/p)/(h) \bigoplus_{q\neq m\in f(n)} s^n R/q \oplus \bigoplus_{i\neq n} \bigoplus_{q\in f(i)} s^i R/q$ then $P_{M(f)} = P_N$.*

Notice to get N from $M(f)$ we simply replaced $s^n R/m$ by $s^n(R/p)/(h)$ and left everything else the same.

Proof. The only prime that contains $p + (h) = \mathrm{ann}((R/p)/(h))$ is m and therefore by [10, Theorem 3.1(a)], $\mathrm{Ass}((R/p)/(h)) = \{m\}$. Writing $m = (p_1, p_2, \ldots, p_k)$ the Krull principal ideal theorem [10, Theorem 10.2] or [16, Chapter 5, Theorem 18] implies that the height of m is at most k and thus since R is a local ring with maximal ideal m, $\dim R \leqslant k$. In particular $\dim R$ is finite, and therefore we can apply Theorem 6.12 to see that $s^n(R/p)/(h) < s^n R/m$ and $s^n R/m < s^n(R/p)/(h)$. It follows that $M(f) < N$ and $N < M(f)$, therefore $P_{M(f)} = P_N$. $\square$

Lemma 7.2. *Suppose R is a local ring with maximal ideal m, and p a prime maximal under m. Let $h \in m - p$ be any element. Suppose $f \in \{\text{perversity functions}\}$ such that $p \notin f(n+1)$.*

$$s^{n+1}R/p\left[\frac{1}{h}\right] \to P_{M(f)}\left(s^{n+1}R/p\left[\frac{1}{h}\right]\right)$$

is injective on H_{n+1}.

Proof. Recall (see Eq. (2.16) and above it) we have a triangle

$$\left(s^{n+1}R/p\left[\frac{1}{h}\right]\right)\langle M(f)\rangle \xrightarrow{i} s^{n+1}R/p\left[\frac{1}{h}\right] \xrightarrow{j} P_{M(f)}\left(s^{n+1}\left(R/p\left[\frac{1}{h}\right]\right)\right)$$
$$\to s\left(s^{n+1}R/p\left[\frac{1}{h}\right]\right)\langle M(f)\rangle.$$

Also $s^{n+1}R/p[\frac{1}{h}]\langle M(f)\rangle \in \overline{C(M(f))}$. Since $p \notin f(n+1)$, for every $q \subset p$ and $i \leqslant n+1$, $q \notin \mathrm{Supp}\, H_i(M(f))$, so Lemma 2.17 implies that $q \notin \mathrm{Supp}\, H_{n+1}((s^{n+1}R/p[\frac{1}{h}])\langle M(f)\rangle)$ and hence $q \notin \mathrm{Ass}\, H_{n+1}(s^{n+1}(R/p[\frac{1}{h}])\langle M(f)\rangle)$. This implies using Lemma 2.3, that $p \notin \mathrm{Ass}\,\mathrm{im}(H_{n+1}(i))$. Since $\mathrm{Ass}\, H_{n+1}(s^{n+1}(R/p[\frac{1}{h}])) = \{p\}$, we get that $H_{n+1}(i) = 0$ and thus from the long exact sequence of the triangle above that $H_{n+1}(j)$ is injective, thus proving the lemma. $\square$

Lemma 7.3. *Suppose R is a local ring with maximal ideal m, and p a prime maximal under m. Let $h \in m - p$ be any element. Suppose $f \in \{\text{perversity functions}\}$ such that $m \in f(n)$, then $P_{M(f)}(s^{n+1}R/p) \cong P_{M(f)}s^{n+1}(R/p[\frac{1}{h}])$.*

Proof. Considering the exact sequence

$$0 \to R/p \xrightarrow{\times h} R/p \to (R/p)/(h) \to 0.$$

We get a triangle

$$s^n(R/p)/(h) \to s^{n+1}R/p \xrightarrow{\times h} s^{n+1}R/p \to s^{n+1}(R/p)/(h).$$

Thus Proposition 3.6(2) implies that $s^{n+1}R/p \xrightarrow{\times h} s^{n+1}R/p$ is a $P_{s^n(R/p)/(h)}$ equivalence. With the N from Lemma 7.1, and also using Lemmas 3.2 and 3.3 we see that $M(f) < N < s^n(R/p)/(h)$. So $s^{n+1}R/p \xrightarrow{\times h} s^{n+1}R/p$ is also a $P_{M(f)}$ equivalence by Proposition 3.4(6). Since $s^{n+1}R/p[\frac{1}{h}]$ is the colimit of such maps, it follows from Proposition 3.6(1) that $s^{n+1}R/p \to s^{n+1}(R/p[\frac{1}{h}])$ is a $P_{M(f)}$ equivalence. Thus, $P_{M(f)}(s^{n+1}R/p) \cong P_{M(f)} \times (s^{n+1}R/p[\frac{1}{h}])$ by Proposition 3.4 and Corollary 3.5. □

Lemma 7.4. *Using notation from the last few lemmas, $R/p[\frac{1}{h}]$ is not a finitely generated R-module.*

Proof. We know that $R/p[\frac{1}{h}] = colim(R/p \overset{\times h}{\to} R/p \xrightarrow{\times h} \cdots)$. Since R/p is an integral domain, each map $\times h$ is injective and since $h \in m \setminus p$, h is not a unit and so $\times h$ is not surjective. These two facts imply that $R/p[\frac{1}{h}]$ is not a finitely generated R-module. □

Lemma 7.5. *Suppose $\overline{C(E)}$ is an aisle and $E = \bigoplus_\alpha E_\alpha$ with each $E_\alpha \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$, in other words E is a direct sum of objects from $D^{\mathrm{b}}_{\mathrm{fg}}(R)$. Then for any $M \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$,*

$$(P_E M) \otimes R_p \cong P_{E \otimes R_p}(M \otimes R_p),$$

in $D(R_p)$, where the $P_{E\otimes R_p}$ is taken in $D(R_p)$.

Proof. Let $\mathcal{S} = \{B \in D(R) \mid B \otimes R_p \in \overline{C(E \otimes R_p)}\}$. In this case we are considering $\overline{C(E \otimes R_p)}$ as a nullity class of R_p-modules. Again as in the proof of Lemma 6.1, the fact that $\_ \otimes R_p$ preserves triangles and colimits, implies that $\mathcal{S}$ is a cocomplete pre-aisle containing E. So $\overline{C(E)} \subset \mathcal{S}$ and thus $\overline{C(E)} \otimes R_p \subset \overline{C(E \otimes R_p)}$.

In the triangle

$$A \to M \to P_E M$$

$A \in \overline{C(E)}$ and tensoring the triangle with R_p we get a triangle in $D(R_p)$

$$A \otimes R_p \to M \otimes R_p \to P_E M \otimes R_p.$$

Also $A \otimes R_p \in \overline{C(E)} \otimes R_p \subset \overline{C(E \otimes R_p)}$ and hence $M \otimes R_p \to P_E M \otimes R_p$ is a $P_{E\otimes R_p}$ equivalence by Proposition 3.6.

By Proposition 3.4, for $k \geqslant 0$, $\mathrm{Hom}_{D(R)}(s^k E, P_E(M)) = 0$. Thus for each α and $k \geqslant 0$, $\mathrm{Hom}_{D(R)}(s^k E_\alpha, P_E(M)) = 0$. Since $\overline{C(E)}$ is an aisle, $P_E M \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$ and Lemma 2.6 then implies that for every α and $k \geqslant 0$, $\mathrm{Hom}_{D(R_p)}(s^k E_\alpha \otimes R_p, P_E M \otimes R_p) = 0$. Since tensor products commute with direct sums and Hom in the first variable turns direct sums into products, we see that for every $k \geqslant 0$, $\mathrm{Hom}_{D(R_p)}(s^k E \otimes R_p, P_E M \otimes R_p) = 0$. Thus by Proposition 3.4(5), $P_E M \otimes R_p$ is $P_{E\otimes R_p}$ local. Thus the lemma follows from Corollary 3.5. □

Proposition 7.6. *Suppose $p' \in \operatorname{Spec} R$ and p a prime maximal under p'. Suppose $f \in$ {perversity functions} such that $\phi(f)$ is an aisle, $p' \in f(n)$, $p' \notin f(n-1)$ and $p \notin f(n+1)$, then $P_{M(f)}(s^{n+1}R/p) \notin D^{\mathrm{b}}_{\mathrm{fg}}(R)$.*

Proof. For our perversity functor $f : \mathbb{Z} \to \mathcal{P}(\operatorname{Spec} R)$ set $f_{p'} : \mathbb{Z} \to \mathcal{P}(\operatorname{Spec} R_{p'})$ to be,

$$f_{p'}(n) = \{q \in \operatorname{Spec} R_{p'} \mid q \in f(n) \text{ and } q \subset p'\}.$$

We see that $f_{p'}(n) = f(n) \cap \operatorname{Spec} R_{p'}$, and also clearly $M(f) \otimes R_{p'} \cong M(f_{p'}) \in D(R_{p'})$. In addition $f_{p'}(n-1) = \emptyset$, $p' \in f(n)$ and $p \notin f(n+1)$.

Since $\phi(f)$ is an aisle and $M(f)$ is a direct sum of objects in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ we can use Lemma 7.5 to see that in $D(R_{p'})$ with $P_{M(f_{p'})}$ also taken in $D(R_{p'})$,

$$\big(P_{M(f)}s^{n+1}R/p\big) \otimes R_{p'} \cong P_{M(f)\otimes R_{p'}}\big(s^{n+1}R/p \otimes R_{p'}\big) \cong P_{M(f_{p'})}\big(s^{n+1}R/p \otimes R_{p'}\big).$$

By Lemmas 7.2, 7.3 and 7.4, the non-finitely generated $R_{p'}$ module $R/p \otimes R_{p'}[\frac{1}{h}]$ injects into $H_{n+1}(P_{M(f_{p'})}(s^{n+1}R/p \otimes R_{p'}))$. Thus it is not finitely generated, and also we see that $H_{n+1}(P_{M(f)}s^{n+1}R/p \otimes R_{p'})$ cannot be finitely generated. Hence $P_{M(f)}s^{n+1}R/p \otimes R_{p'} \notin D^{\mathrm{b}}_{\mathrm{fg}}(R_{p'})$ and so $P_{M(f)}s^{n+1}R/p \notin D^{\mathrm{b}}_{\mathrm{fg}}(R)$. □

The next result will imply that the $\phi(f)$ of the last proposition can't actually be an aisle.

Theorem 7.7. *Suppose $E \in D(R)$ and $\mathcal{D} \subset D(R)$ is a full triangulated subcategory. If for every $M \in \mathcal{D}$, $P_E M \in \mathcal{D}$ then the nullity class $\mathcal{A} = \overline{C(E)} \cap \mathcal{D}$ is an aisle. If there exists a set $\{E(\alpha)\}_{\alpha<\lambda}$ of objects in $\mathcal{A}$ such that $\bigoplus_{\alpha<\lambda} E(\alpha) < E$, then the converse is also true.*

Proof. Suppose that for every $M \in \mathcal{D}$, $P_E M \in \mathcal{D}$. Using Theorem 2.15, we see that $M\langle E\rangle$ is always in $\overline{C(E)}$ and looking at the distinguished triangle of Eq. (2.16), we see that $M\langle E\rangle \in \mathcal{D}$ as well. Thus the functor $M \mapsto M\langle E\rangle$ gives the required right adjoint to the inclusion $\mathcal{A} \subset \mathcal{D}$ and so $\mathcal{A}$ is an aisle.

Now suppose $\mathcal{A} = \overline{C(E)} \cap \mathcal{D}$ is an aisle, and let $M \in \mathcal{D}$. It is well known that, see [1, Proposition 1.1] for example for a proof, that we have a triangle in $\mathcal{D}$

$$M\langle\mathcal{A}\rangle \to M \to P_{\mathcal{A}}M \to sM\langle\mathcal{A}\rangle$$

(see Eq. (2.12) and above it) such that:

a) $M\langle\mathcal{A}\rangle \in \mathcal{A} \subset \overline{C(E)}$.
b) $P_{\mathcal{A}}M \in \mathcal{A}^{\perp}$.

By Proposition 3.6(2), a) implies that $M \to P_{\mathcal{A}}M$ is a P_E equivalence.

Statement b) above says that for every $X \in \mathcal{A}$, $\mathrm{Hom}(X, P_{\mathcal{A}}M) = 0$. In particular for every α and $k \geqslant 0$, $s^k E(\alpha) \in \mathcal{A}$, and so $\mathrm{Hom}(s^k E(\alpha), P_{\mathcal{A}}M) = 0$. Thus for every $k \geqslant 0$,

$$\mathrm{Hom}\left(s^k \bigoplus_{\alpha<\lambda} E(\alpha), P_{\mathcal{A}}M\right) = \prod_{\alpha<\lambda} \mathrm{Hom}\big(s^k E(\alpha), P_{\mathcal{A}}M\big) = 0,$$

and so $P_{\mathcal{A}}M$ is $P_{\bigoplus_{\alpha<\lambda} E(\alpha)}$ local and thus since $\bigoplus_{\alpha<\lambda} E(\alpha) < E$, P_E local by Proposition 3.4(6). So by Corollary 3.5, $P_E M \cong P_{\mathcal{A}}M \in \mathcal{D}$. $\square$

If $\mathcal{D}$ has a set of objects we can always take $\{E(\alpha)\} = \mathcal{A}$. Then $P_{\mathcal{A}} = P_{\bigoplus_{\alpha<\lambda} E(\alpha)}$, so restricted to $\mathcal{D}$, $P_E = P_{\bigoplus_{\alpha<\lambda} E(\alpha)}$. Thus the converse always holds if $\mathcal{D}$ has a set of objects. The condition that $\bigoplus_{\alpha<\lambda} E(\alpha) < E$ is needed since something could be in $\overline{C(E)}^{\perp}$ when restricted to maps in a smaller category, like $\mathcal{D}$, but no longer in $\overline{C(E)}^{\perp}$ in $D(R)$. The same problem comes up when you try to construct cohomological localizations in the category of spectra.

Corollary 7.8. *For $f \in$ {perversity functions}, $N(f)$ is an aisle if and only if for every $S \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$, $P_{M(f)}S \in D^{\mathrm{b}}_{\mathrm{fg}}(R)$.*

Proof. By definition $N(f) = \overline{C(M(f))} \cap D^{\mathrm{b}}_{\mathrm{fg}}(R)$ and $M(f) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in f(n)} s^n R/p$, so in particular $\bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in f(n)} s^n R/p < M(f)$ and each $s^n R/p \in N(f)$. Thus the corollary follows from the theorem. $\square$

7.1. Aisles should correspond to comonotone perversities

Recall from 4.2, a perversity function $f : \mathbb{Z} \to \operatorname{Spec} R$ is comonotone if whenever $p' \in f(n)$ and p is maximal under p', then $p \in f(n+1)$.

Theorem 7.9. *If $\mathcal{A}$ is an aisle in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, then its perversity function $\phi(\mathcal{A}) : \mathbb{Z} \to \operatorname{Spec} R$ is comonotone.*

Proof. By Theorem 5.6, $N\phi(\mathcal{A}) = \mathcal{A}$ and so $P_{\mathcal{A}} = P_{M(\phi(\mathcal{A}))}$. Suppose $p' \in \phi(\mathcal{A})(n)$, $p' \notin \phi(\mathcal{A})(n-1)$ and p is maximal under p'. If $p \notin \phi(\mathcal{A})(n+1)$ then Proposition 7.6 would imply that $P_{M(\phi(\mathcal{A}))} s^{n+1} R/p \notin D^{\mathrm{b}}_{\mathrm{fg}}(R)$ and by Corollary 7.8 that would imply that $\mathcal{A}$ is not an aisle. So we must have that $p \in \phi(\mathcal{A})(n+1)$. Since $\phi(\mathcal{A})$ is increasing as well we easily see that $\phi(\mathcal{A})$ is comonotone. $\square$

We conjecture that this is in fact the only extra thing that is needed for a perversity function to correspond to an aisle in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$.

Conjecture 7.10. *For a noetherian ring R, if a perversity function $f : \mathbb{Z} \to \operatorname{Spec} R$ is comonotone then its nullity class $N(f) \subset D^{\mathrm{b}}_{\mathrm{fg}}(R)$ is an aisle.*

The converse of this is Theorem 7.9 and by Corollary 6.14 or Theorem 5.6, all aisles in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$ are of this form. So proving this conjecture would complete the classification of t-structures in $D^{\mathrm{b}}_{\mathrm{fg}}(R)$. As mentioned before if R has a dualizing complex this conjecture is a special case of the main theorem of [4].

Example 7.11. Quasi-coherent complexes are really the right ones to work with in this case since there are additional restrictions for having a t-structure in $D_{\mathrm{perf}}(R)$. For example consider again $D_{\mathrm{perf}}(\mathbb{Z}/4)$ and $f:\mathbb{Z}\to \operatorname{Spec}\mathbb{Z}/4=\{(2)\}$ given by

$$f(i)=\begin{cases}\emptyset i\leqslant 0,\\ \{(2)\} & i>0.\end{cases}$$

Then $M(f)=\bigoplus_{i>0} s\mathbb{Z}/2$, $N(f)$ is just all complexes with homology concentrated in possitive degrees and $P_{M(f)}$ is just truncation. Letting A be the complex

$$A_i=\begin{cases}\mathbb{Z}/4 & i=0,1,\\ 0 & \text{else}\end{cases}$$

and $d:A_1\to A_0$ be multiplication by 2, we can see that $P_{M(f)}A=\mathbb{Z}/2$, but $\mathbb{Z}/2\notin D_{\mathrm{perf}}(\mathbb{Z}/4)$ since any resolution of it has infinite length. Also $sA\in \overline{C(M(f)}\cap D_{\mathrm{perf}}(R)$, and $sA<s\mathbb{Z}/2$, so we get by Theorem 7.7 that $\overline{C(M(f))}\cap D_{\mathrm{perf}}(R)$ is not an aisle.

8. A class of t-structures in $D(\mathbb{Z})$

In this section we show that the t-structures in $D(\mathbb{Z})$ do not form a set but rather a proper class (Corollary 8.4). The same proof shows that the nullity classes in spectra and in topological spaces do not form a set. Similarly the t-structures in the triangulated category of spectra do not form a set. These results follows easily from some nice, and more difficult, examples of Shelah [19].

There are two main reasons we chose to exhibit this result. The first reason is to show that it is unreasonable to expect a nice classification of t-structures or nullity classes in $D(R)$. The second related reason is to contrast with what happens in the case of localizing subcategories in $D(R)$. If we demand that our nullity classes are also closed under taking desuspensions, we get a localizing category in $D(R)$. Neeman [17] showed that these are in 1-1 correspondence with subsets of Spec R, so the situation is only slightly more complicated than for thick subcategories of $D_{\mathrm{perf}}(R)$. So going from something with some finiteness conditions, $D_{\mathrm{perf}}(R)$, to infinite things, $D(R)$, only increases complexity slightly. However the situation for nullity classes is much different. In the finite case, $D^{\mathrm{b}}_{\mathrm{fg}}(R)$, we have a classification more or less in terms of increasing sequences of thick subcategories; when we move to $D(R)$ though, we completely lose control, we might have a proper class of nullity classes, and even with the extra condition required for a t-structure still have a proper class.

Definition 8.1. A system $\{A_\alpha\}_{\alpha\in Y}$ of distinct abelian groups is called a rigid system if $\alpha\neq\beta$ implies that $\operatorname{Hom}(A_\alpha, A_\beta)=0$.

In [19], Shelah proved the following:

Theorem 8.2. *[19] For every cardinal λ there is a rigid system of abelian groups $\{A_\alpha\}_{\alpha\in 2^\lambda}$ such that $|A_\alpha|=\lambda$.*

Proposition 8.3. *Consider a rigid system $\{A_\alpha\}_{\alpha\in Y}$ of abelian groups. If $\alpha\neq\beta$ then in $D(\mathbb{Z})$, $P_{A_\alpha}A_\beta=A_\beta\neq 0$ hence $A_\alpha\nless A_\beta$.*

Proof. Suppose $\alpha \neq \beta$. Since $\{A_\alpha\}$ is a rigid system, $\mathrm{Hom}(A_\alpha, A_\beta) = 0$ and since $H_i(A_\alpha) = 0$ for $i < 0$ and $H_i(A_\beta) = 0$ for $i > 0$, $\mathrm{Hom}(s^i A_\alpha, A_\beta) = 0$ for all $i > 0$. Thus $P_{A_\alpha} A_\beta = A_\beta \neq 0$. That $A_\alpha \not< A_\beta$ then follows directly from the definition of $<$. □

Corollary 8.4. *The class of t-structures, and hence also the class of nullity classes, in $D(\mathbb{Z})$ do not form a set.*

Proof. For any cardinal λ let $\{A_\alpha\}_{\alpha \in 2^\lambda}$ be the rigid system of abelian groups of Theorem 8.2. To each A_α using Theorem 2.15 we associate the aisle $\overline{C(A_\alpha)}$. By Proposition 8.3, if $\alpha \neq \beta$ then $A_\alpha \not< A_\beta$ and hence by Proposition 3.3 $\overline{C(A_\alpha)} \neq \overline{C(A_\beta)}$. Thus the aisles $\{\overline{C(A_\alpha)}\}_{\alpha \in 2^\lambda}$ are all distinct, which means that the t-structures $(\overline{C(A_\alpha)}, s\overline{C(A_\alpha)}^\perp)$ are also distinct. So we see that there are at least 2^λ distinct t-structures. Since λ is arbitrary the proof is complete. □

In the category of spectra by results of Bousfield, homological localizations are of the form P_A. It was shown by Ohkawa [18] that this subclass of localizations do form a set. It is unknown if all localizations of the form P_A which are stable under desuspension form a set. The next theorem shows that if we take all localizations of the form P_A without assuming they are closed under suspension then we do not get a set.

Theorem 8.5. *The class of t-structures, and hence also the class of nullity classes, in spectra do not form a set. Similarly the class of nullity classes in spaces do not form a set.*

Proof. We really only give an outline of the proof. Those initiated to the yoga of P_A can easily fill in the details. Recall that $K(G, n)$ is the Eilenberg–Mac Lane spectrum (or space) with homotopy groups G concentrated in dimension n. The functors P_E have been constructed in spaces and spectra, for example see [9] and [11]. For spectra and any E, $\overline{C(E)}$ is an aisle. Then for a rigid system $\{A_\alpha\}_{\alpha \in 2^\lambda}$ of abelian groups, all the nullity classes (and t-structures if we are working in spectra) $\overline{C(K(A_\alpha, n))}$ are distinct for the same reasons as above. Since λ is arbitrary this means that there is not a set of them. □

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